Faces of homogeneous cones and its applications

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June 02, 2026
SIAM OP 26

Faces of $\mathcal{S}^n_+$ (positive semidefinite matrices)

Theorem (Folklore, Barker and Carlson'75)

Let $\mathcal{F} \subseteq \mathcal{S}^n_+$ be a face. Then, there exists an orthogonal Q such that

$$Q\mathcal{F}Q^* = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \mid A \in \mathcal{S}^r_+ \right\}$$

Let $R := Q^* \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} Q$ and $\mathcal{P}(X) := RXR^*$

$$\mathcal{P}(\mathcal{S}^n_+) = \mathcal{F}, \qquad \mathcal{P}^2 = \mathcal{P}$$

Faces of $\mathcal{S}^n_+$ are **projectionally exposed** and linearly isomorphic to some $\mathcal{S}^r_+$ for $r \leq n$.

- Computing Q is easy: it is enough to look at the kernel of any $x \in \operatorname{ri} \mathcal{F}$.

Projectional exposedness

$\mathcal{F}$: face of $\mathcal{K}$

1. $\mathcal{F}$ is **projectionally exposed** $\overset{\text{def}}{\iff}$ $\exists P$ linear, $P(\mathcal{K}) = \mathcal{F}$, $P^2 = P$.
   - i.e., $\mathcal{F}$ is a projection of $\mathcal{K}$.
2. $\mathcal{K}$ is **projectionally exposed** $\overset{\text{def}}{\iff}$ every face is projectionally exposed.
3. Symmetric cones and polyhedral cones are projectionally exposed.

Borwein and Wolkowicz. *Regularizing the abstract convex program. JMAA*, 1981.

Sung and Tam. *A study of projectionally exposed cones. LAA, 1990.*

L. *Amenable cones: error bounds without constraint qualifications.* Math. Prog. 2021.

How is this helpful?

$$\max\{b^T y \mid c - \sum_{i=1}^{m} A_i y_i \in \mathcal{K}\} \overset{\text{Facial reduction}}{\Longrightarrow} \max\{b^T y \mid c - \sum_{i=1}^{m} A_i y_i \in \mathcal{F}\}$$

- Regularizes but does not reduce the size of the problem. (c and A_i are unchanged!)

Much better to do this:

$$\begin{aligned} \max \quad & b^T y \\ & P(c) - \sum_{i=1}^{m} P(A_i) y_i \in \mathcal{F} \\ & Q(c) - \sum_{i=1}^{m} Q(A_i) y_i = 0, \end{aligned}$$

where $Q = I - P$, $P^2 = P$, $P(\mathcal{K}) = \mathcal{F}$.

- P allows us to rewrite the problem over the span of $\mathcal{F}$.

Homogeneous cones

Today's goals

- Describe the facial structure of homogeneous cones.
- Prove they are projectionally exposed.

- $\mathcal{K}$: full-dimensional convex cone
- $A \in \mathrm{Aut}(\mathcal{K}) \Leftrightarrow A(\mathcal{K}) = \mathcal{K}$ and A is linear.

$\mathcal{K}$ is **homogeneous** $\Leftrightarrow \forall x, y \in \operatorname{int} \mathcal{K}, \exists A \in \mathrm{Aut}(\mathcal{K}), Ax = y$.

- Examples: $\mathbb{R}^n_+$, $\mathcal{S}^n_+$, **certain cones of sparse PSD matrices** and many others.
- Theoretical foundations were laid by Gindikin, Rothaus, Vinberg and others in the 60s.

Gouveia, Ito and L. *Faces of homogeneous cones and applications to homogeneous chordality.* arxiv:2501.09581.

Some history

- Truong and Tunçel (MP'04) showed **facial exposedness**.
- Chua and Tunçel (MP'08) showed **niceness**.
- L, Roschina and Saunderson (SIOPT'22) showed they are **amenable**.
- Here we will prove they are **projectionally exposed**.

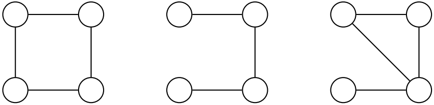
If you like Jordan Algebras you are in for a treat! If not, at least it is just 25 min.

Homogeneous chordal cones

- $G = (V, E)$: a graph

$$\mathcal{S}_+(G) := \{x \in \mathcal{S}^n_+ \mid x_{ij} = 0 \text{ for all } i \neq j \text{ such that } \{i,j\} \notin E\}$$

- Chordal graph: Every cycle on 4 or more vertices has a chord.
- Homogeneous chordal: Chordal and no induced subgraph is a path on 4 vertices.



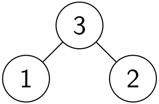
(a) Not chordal (b) Chordal (c) Homogeneous chordal

Theorem (Ishi'13)

G is homogeneous chordal if and only if $\mathcal{S}_+(G)$ is a homogeneous cone.

See also Tunçel and Vandenberghe (Acta Numerica '23).

Example



$$\mathcal{S}_+(G) = \left\{ A = \begin{pmatrix} x_{11} & 0 & x_{13} \\ 0 & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{pmatrix} \mid A \succeq 0 \right\}$$

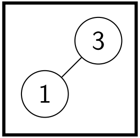
Some questions:

- Connection between **faces** of $\mathcal{S}_+(G)$ and **induced subgraphs of** G?
- Is $\mathcal{S}_+(G)$ projectionally exposed?
- What are its faces?

Principal faces

- $G = (V, E)$: Homogeneous chordal.
- H: induced subgraph of G.
- $\mathcal{F}(H) : \{x \in \mathcal{S}_+(G) \mid x_{ii} = 0, i \notin H\}$ (**principal faces**)

Example:



$$\mathcal{F}(H) = \left\{ A = \begin{pmatrix} x_{11} & 0 & x_{13} \\ 0 & 0 & 0 \\ x_{13} & 0 & x_{33} \end{pmatrix} \mid A \succeq 0 \right\}.$$

- Principal faces are **projectionally exposed**.
- $\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \mapsto \begin{pmatrix} * & 0 & * \\ 0 & 0 & 0 \\ * & 0 & * \end{pmatrix}$
- Finitely many principal faces, though.

Strategy

- $\mathcal{F}(H)$ is **projectionally exposed (easy)**
- Show that every face $\mathcal{F}$ satisfies $Q(\mathcal{F}) = \mathcal{F}(H)$ for some H and some $Q \in \mathrm{Aut}(\mathcal{S}_+(G))$ (**hard**).

A preliminary: Lower triangular matrices

$$\mathcal{L}_+(G) = \left\{ \begin{pmatrix} l_{11} & 0 & 0 \\ 0 & l_{22} & 0 \\ l_{13} & l_{23} & l_{33} \end{pmatrix} \mid l_{ii} \geq 0 \right\}$$

$$\mathcal{L}_{++}(G) = \left\{ \begin{pmatrix} l_{11} & 0 & 0 \\ 0 & l_{22} & 0 \\ l_{13} & l_{23} & l_{33} \end{pmatrix} \mid l_{ii} > 0 \right\}$$

- $\mathcal{L}_+(G)$: "sparse lower triangular matrices with nonnegative diagonal".
- $\mathcal{L}_{++}(G)$: "sparse lower triangular matrices with positive diagonal".

Two basic facts:

$\mathcal{L}_{++}(G)$ acts transitively on $\mathrm{int}(\mathcal{S}_+(G))$ via $x \mapsto lxl^*$.

$\mathrm{int}(\mathcal{S}_+(G)) = \{ll^* \mid l \in \mathcal{L}_{++}(G)\}$ and the decomposition $x = ll^*$ is unique for $x \in \mathrm{int}(\mathcal{S}_+(G))$.

Strategy

Idea

- $\mathcal{F}(H)$ is **projectionally exposed (easy)**
- Show that every face $\mathcal{F} \trianglelefteq \mathcal{S}_+(G)$ satisfies $Q(\mathcal{F}) = \mathcal{F}(H)$ for some H and some $Q \in \mathrm{Aut}(\mathcal{S}_+(G))$ (**hard**).

Given $\mathcal{F}$ we ind H via **Generalized Cholesky factors**.

- First, we order the vertices appropriately. (TPEO)
- Let $x \in \mathrm{ri}\,\mathcal{F}$. There may be several l satisfying $x = ll^*$ and $l \in \mathcal{L}_+(G)$.
- A unique $l \in \mathcal{L}_+(G)$ satisfying $l_{ii} = 0 \Rightarrow l_{ji} = 0\,\forall j$ and $x = ll^*$ exists. (**Generalized Cholesky decomposition**)
- H is the subgraph induced by $\{i \mid l_{ii} \neq 0\}$.

Faces of $\mathcal{S}_+(G)$ - GIL'25

Let

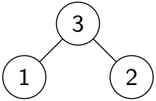
- $G = (V, E)$: homogeneous chordal graph with a TPEO
- $\mathcal{F}$: face of $\mathcal{S}_+(G)$
- $x \in \mathrm{ri}\,\mathcal{F}$

Write $x = ll^*$ with a **generalized Cholesky factor** l, let $J := \{i \mid l_{ii} \neq 0\}$

- $\mathcal{F}$ is linearly to isomorphic $\mathcal{F}(H)$, where H is the subgraph induced by J.
- $\mathcal{F}$ is projectionally exposed.

- $\mathcal{F}(H)$: principal face obtained by zeroing the rows/columns corresponding to indices that are not vertices of H.

An example



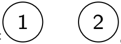
$$x = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

> What is the minimal face $\mathcal{F}$ of $\mathcal{S}_+(G)$ containing x? Which H satisfies $\mathcal{F} \cong \mathcal{F}(H)$?

Reminder: If $\mathcal{F} \trianglelefteq \mathcal{K}$, $\mathcal{F} = \mathcal{F}_{\mathsf{min}}(x, \mathcal{K}) \iff x \in \mathrm{ri}\,\mathcal{F}$.

1) Decompose: $x = ll^*$ for

$$l := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$J = \{i \mid l_{ii} \neq 0\} = \{1, 2\}$. $\mathcal{F}$ **is isomorphic to the face generated by** $H =$ .

An example (cont.)

2) Find the automorphism: find $t \in \mathcal{L}_{++}$ such that

$$tI = e_J, \text{ where } e_J := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

One possible solution is $t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$.

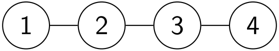
$Q_t : y \mapsto tyt^*$ is an automorphism of $\mathcal{S}_+(G)$ mapping x to e_J and

$$Q_I(\mathcal{F}) = \mathcal{F}(H),$$

where $\mathcal{F}(H)$ is the principal face associated to H.

- $\hat{I} = t^{-1} e_I t$ is such that $Q_{\hat{I}} : y \mapsto \hat{I} y \hat{I}^*$ projects $\mathcal{S}_+(G)$ onto $\mathcal{F}$.

Failure for chordal graphs



Let $\mathcal{F} := \{x \in \mathcal{S}_+(G) \mid (1,1,1,1) \in \ker x\}$.

- $\mathcal{F} \cong \mathbb{R}^3_+$ and contains a rank 3 matrix. $\mathcal{F}$ **is polyhedral**.
- Induced subgraphs with polyhedral principal faces: $\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,3\}, \{1,4\}, \{2,4\}$. None of them contain rank 3 matrices.

There is no principal face of $\mathcal{S}_+(G)$ linearly isomorphic to $\mathcal{F}$!

T-algebra and homogeneous cones

$\mathcal{A} = \bigoplus_{i,j=1}^{r} \mathcal{A}_{ij}$: a matrix algebra with involution * + extra conditions. (T-Algebra)

$$\mathcal{L} := \bigoplus_{1 \le j \le i \le r} \mathcal{A}_{ij},$$
$$\mathcal{L}_+ := \{a \in \mathcal{L} \mid a_{ii} \ge 0\},$$
$$\mathcal{L}_{++} := \{a \in \mathcal{L} \mid a_{ii} > 0\}$$

$$\mathcal{K}(\mathcal{A}) := \{ll^* \mid l \in \mathcal{L}_{++}\}$$
$$\mathrm{cl}\,\mathcal{K}(\mathcal{A}) = \{ll^* \mid l \in \mathcal{L}_{+}\}$$

Vinberg'65

- $\mathcal{K}(\mathcal{A})$ is a open homogeneous convex cone in $\mathcal{H}(\mathcal{A}) := \{a = a^* \mid a \in \mathcal{A}\}$
- $\mathcal{K}$ is a pointed open homogeneous convex cone $\Leftrightarrow \exists \mathcal{A}$, s.t., $\mathcal{K} \cong \mathcal{K}(\mathcal{A})$.

Principal subalgebras and faces

Let

- $I \subseteq \{1, \dots, r\}$.
- $\mathcal{A}_I := \{a \in \mathcal{A}_I \mid a_{ik} = a_{ki} = 0, \forall i \in I, \forall k \in \{1, \dots, r\}\}$
- $\mathcal{A}_I$ is the subalgebra where the rows and columns in I are removed.

- $\mathcal{A}_I$ is a T-algebra.
- $\operatorname{cl} \mathcal{K}(\mathcal{A}_I)$ is a **projectionally exposed face** of $\operatorname{cl} \mathcal{K}(\mathcal{A})$.

The same game as before

Every face is mapped to a principal face via an automorphism.

Faces of homogeneous convex cones - GIL'25

Let $\operatorname{cl}\mathcal{K}(\mathcal{A})$ be a homogeneous convex cone and

- $\mathcal{F}$ a face of $\operatorname{cl}\mathcal{K}(\mathcal{A})$.
- $x \in \operatorname{ri}\mathcal{F}$

Write the **generalized Cholesky decomposition** $x = ll^*$ $(l \in \mathcal{L}_+)$ and let $I := \{i \mid l_{ii} = 0\}$.

1. There is an automorphism Q with $Q(\mathcal{F}) = \operatorname{cl}\mathcal{K}(\mathcal{A}_I)$.
2. $\mathcal{F}$ is a homogeneous cone of rank $r - |I|$.
3. $\mathcal{F}$ is projectionally exposed.

As I bid my farewell...

- Homogeneous cones are projectionally exposed.
- Each face $\mathcal{F}$ is linearly isomorphic to one of the 2^r principal face.
- The generalized Cholesky decomposition of $x \in \operatorname{ri} \mathcal{F}$ gives all the information.

More stuff in the paper!

- Conjugate faces
- PSD completion problems
- Other algebraic approaches.

Gouveia, Ito and L. *Faces of homogeneous cones and applications to homogeneous chordality.* arxiv:2501.09581.