Error Bounds and Facial Residual Functions for Conic Linear Programs

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based on works with Scott B. Lindstrom, Tianxiang Liu,
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March 14th, 2022
OWOS

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{subject to} \quad & h(x) = 0 \end{aligned}$$

- Suppose I use my favourite solver and obtain x^*.
- The solver tells me that the KKT conditions are satisfied to $\epsilon = 10^{-6}$.
- It also tells me that $\|h(x^*)\| \leq 10^{-7}$.

Question 1

Is x^* close to the set of **optimal** solutions?

Question 2

Is x^* close to the set of **feasible** solutions?

Distance to a set C: $\operatorname{dist}(x, C) := \inf_{y \in C} \|x - y\|$.

An example by Sturm (SIOPT'00)

$$\begin{aligned} \text{find} \quad & x \in S^3 \\ \text{subject to} \quad & \begin{pmatrix} x_{11} & x_{33} & x_{13} \\ x_{33} & 0 & 0 \\ x_{13} & 0 & x_{33} \end{pmatrix} \succeq 0. \end{aligned}$$

- Feasible set: matrices $\begin{pmatrix} x_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with $x_{11} \geq 0$.

An example by Sturm

Let $\epsilon > 0$

$$x_\epsilon := \begin{pmatrix} 3 & \sqrt{\epsilon} & \sqrt[4]{\epsilon} \\ \sqrt{\epsilon} & \epsilon & 0 \\ \sqrt[4]{\epsilon} & 0 & \sqrt{\epsilon} \end{pmatrix}$$

- The constraints are "$x_{22} = 0$", "$x_{12} = x_{33}$" and "$x \in \mathcal{S}^3_+$".
- Suppose we measure the violation of constraints by x using

$$\mathrm{Res}(x) := [x_{22}^2 + (x_{12} - x_{33})^2 + \max\{-\lambda_{\min}(x), 0\}^2]^{1/2}$$

($\mathrm{Res}(x) = 0 \Leftrightarrow x$ is feasible.) x_ϵ does not seem a bad point:

$$\mathrm{Res}(x_\epsilon) = \epsilon$$

But...

$$\mathrm{dist}\,(x_\epsilon, \mathrm{Feas}) \geq \sqrt[4]{\epsilon}.$$

If $\epsilon = 10^{-4}$, we have $\mathrm{Res}(x_\epsilon) = 10^{-4}$, but $\mathrm{dist}\,(x_\epsilon, \mathrm{Feas}) \geq 0.1$.

$$\min_x \quad f(x)$$
$$\text{subject to} \quad h(x) = 0$$

- Suppose I use my favourite solver and obtain x^*.
- The solver tells me that the KKT conditions are satisfied to $\epsilon = 10^{-6}$.
- It also tells me that $\|h(x^*)\| \leq 10^{-7}$.

Question 1

Is x^* close to the set of **optimal** solutions?

Question 2

Is x^* close to the set of **feasible** solutions?

Answer: **Not necessarily!** Also $\mathrm{Res}(x_\epsilon) \to 0$ does not imply $\mathrm{dist}\,(x_\epsilon, \mathrm{Feas}) \to 0$...

Conclusions

- Using solvers, we input the constraints one by one: $h_1(x) = 0, \ldots, h_n(x) = 0,\ g_1(x) \leq 0, g_2(x) \leq 0, \ldots, g_m(x) \leq 0.$
- Solvers can only compute the residuals with respect the g_i and h_j. (**Backward error**)
  - Some measure of error using $|h_j(x)|$, $\max\{g_i(x), 0\}$, or similar quantities are used
- The **true** distance to the feasible region is almost never computable. (**Forward error**)

Backward Error: $\mathrm{Res}(x) := [x_{22}^2 + (x_{12} - x_{33})^2 + \max\{-\lambda_{\min}(x), 0\}^2]^{1/2}$
Forward Error: $\mathrm{dist}\,(x, \mathrm{Feas})$.

Key point

Forward error $\neq O($**Backward Error**$)$

- The same phenomenon happens for optimal sets: small KKT residual $\not\Rightarrow$ the point is close to the optimal set.

What next?

Error bounds provide relations between **Forward error** and **Backward error**.

In this talk: error bounds for problems involving cones.

Feasibility problems over convex cones

Consider the following *feasibility problem over a convex cone* $\mathcal{K}$.

$$\begin{array}{rl} \text{find} & x \\ \text{subject to} & x \in (\mathcal{L} + a) \cap \mathcal{K} \end{array}$$

- $\mathcal{K}$: closed convex cone contained in some space $\mathcal{E}$.
- $\mathcal{L}$: subspace contained in $\mathcal{E}$.
- $a \in \mathcal{E}$.

($\mathcal{L} + a$ is an affine space)

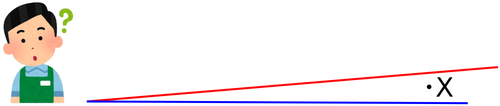
Motivation

Let $\|\cdot\|$ be the Euclidean norm and fix $x \in \mathcal{E}$.

$$\operatorname{dist}(x, \mathcal{L}+a) = \inf\{\|x-y\| \mid y \in \mathcal{L}+a\}$$
$$\operatorname{dist}(x, \mathcal{K}) = \inf\{\|x-y\| \mid y \in \mathcal{K}\}$$
$$\operatorname{dist}(x, (\mathcal{L}+a)\cap\mathcal{K}) = \inf\{\|x-y\| \mid y \in (\mathcal{L}+a)\cap\mathcal{K}\}$$

Fundamental question

Can we estimate $\operatorname{dist}(x, (\mathcal{L}+a)\cap\mathcal{K})$ using $\operatorname{dist}(x, \mathcal{L}+a)$ and $\operatorname{dist}(x, \mathcal{K})$?



- **Backward error**: $\operatorname{dist}(x, \mathcal{L}+a) + \operatorname{dist}(x, \mathcal{K})$
- **Forward error**: $\operatorname{dist}(x, (\mathcal{L}+a)\cap\mathcal{K})$

Hölderian error bounds

C_1, C_2: closed convex sets. $C := C_1 \cap C_2$

Definition (Hölderian error bound)

C_1, C_2 satisfy a **Hölderian error bound** $\overset{\text{def}}{\iff}$ for every bounded set B there exist $\theta_B > 0$, $\gamma_B \in (0,1]$ such that

$$\operatorname{dist}(x, C) \leq \theta_B (\operatorname{dist}(x, C_1) + \operatorname{dist}(x, C_2))^{\gamma_B} \quad \forall\, x \in B.$$

If $\gamma_B = \gamma \in (0,1]$ for all B, the bound is **uniform**. If the bound is uniform with $\gamma = 1$, we call it a **Lipschitzian** error bound.

- $\operatorname{ri} C_1 \cap \operatorname{ri} C_2 \neq \emptyset \Rightarrow$ Lipschitzian
- C_1, C_2 are polyhedral $\Rightarrow$ Lipschitzian (Hoffman's Lemma)
- C_1: polyhedral, $(\operatorname{ri} C_2) \cap C_1 \neq \emptyset \Rightarrow$ Lipschitzian
- C_1: affine space, C_2: PSD cone $\Rightarrow$ Uniform Hölderian (Sturm's error bound, SIOPT'00)

Beyond Sturm's error bound

Today's goals

- Prove error bounds for general cones beyond $\mathcal{S}^n_+$ as **tightly** as possible.

Scott B. Lindstrom; L. and Ting Kei Pong
Error bounds, facial residual functions and applications to the exponential cone
arXiv:2010.16391

Scott B. Lindstrom; L. and Ting Kei Pong
Tight error bounds and facial residual functions for the p-cones and beyond
arXiv:2109.11729

L.
Amenable cones: error bounds without constraint qualifications.
Mathematical Programming, 186:1–48, 2021. (arxiv:1712.06221)

Not gonna lie, these papers are long...
But they are (30%–50% **framework**) + **computation of examples**.

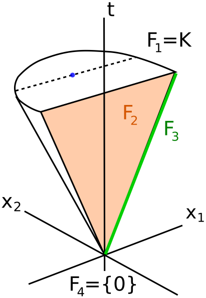
Review of faces 

- $\mathcal{K}$: closed convex cone
- $\mathcal{F} \subseteq \mathcal{K}$: closed convex cone

Definition (Face of a cone)

$\mathcal{F}$ is a face of $\mathcal{K}$ $\Leftrightarrow$ if $x + y \in \mathcal{F}$, with $x, y \in \mathcal{K}$, then $x, y \in \mathcal{F}$.

If $\mathcal{F} \subseteq \mathcal{K}$ is a face, we write $\mathcal{F} \trianglelefteq \mathcal{K}$.



Ingredient 1 - Error bounds under a constraint qualification

$$\begin{aligned} &\text{find} \quad x \\ &\text{subject to} \quad x \in (\mathcal{L} + a) \cap \mathcal{K} \end{aligned} \tag{CFP}$$

Proposition (An error bound for when a face satisfying a CQ is known)

Let $\mathcal{F} \trianglelefteq \mathcal{K}$ be such that

- (a) $\mathcal{F} \cap (\mathcal{L} + a) = \mathcal{K} \cap (\mathcal{L} + a)$
- (b) $(\operatorname{ri} \mathcal{F}) \cap (\mathcal{L} + a) \neq \emptyset$

Then, for every bounded set B, there exists $\kappa_B > 0$ such that

$$\operatorname{dist}(x, \mathcal{K} \cap (\mathcal{L} + a)) \leq \kappa_B(\operatorname{dist}(x, \mathcal{F}) + \operatorname{dist}(x, \mathcal{L} + a)), \qquad \forall x \in B.$$

It is not an error bound with respect to $\mathcal{L} + a$ and $\mathcal{K}$, but it is close.

General strategy

Goal: We want to bound $\operatorname{dist}(x, (\mathcal{L} + a) \cap \mathcal{K})$ using $\operatorname{dist}(x, \mathcal{L} + a)$ and $\operatorname{dist}(x, \mathcal{K})$.

1. Find $\mathcal{F}$ such that
   a. $\mathcal{F} \cap (\mathcal{L} + a) = \mathcal{K} \cap (\mathcal{L} + a)$
   b. $(\operatorname{ri} \mathcal{F}) \cap (\mathcal{L} + a) \neq \emptyset$

   Therefore,

$$\operatorname{dist}(x, \mathcal{K} \cap (\mathcal{L} + a)) \leq \kappa_B(\operatorname{dist}(x, \mathcal{F}) + \operatorname{dist}(x, \mathcal{L} + a)), \quad \forall x \in B. \tag{1}$$

2. Upper bound $\operatorname{dist}(x, \mathcal{F})$ using $\operatorname{dist}(x, \mathcal{K})$ and $\operatorname{dist}(x, \mathcal{L} + a)$.
3. Plug the upper bound in (1).

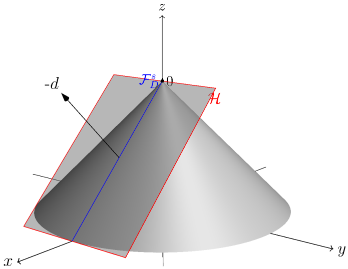
How to find $\mathcal{F}$?

We want $\mathcal{F}$ such that

- (a) $\mathcal{F} \cap (\mathcal{L} + a) = \mathcal{K} \cap (\mathcal{L} + a)$
- (b) $(\mathrm{ri}\,\mathcal{F}) \cap (\mathcal{L} + a) \neq \emptyset$

Idea:

1. Let $\mathcal{F}_1 = \mathcal{K}$ and $i \leftarrow 1$.
2. If $(\mathcal{L} + a) \cap \mathrm{ri}\,\mathcal{F}_i \neq \emptyset$, we are done.
3. If $(\mathcal{L} + a) \cap \mathrm{ri}\,\mathcal{F}_i = \emptyset$, we invoke a separation theorem.
    - There exists $z_i \in \mathcal{F}_i^* \setminus \mathcal{F}_i^\perp$ and $z_i \in \mathcal{L}^\perp \cap \{a\}^\perp$.
    - Let $\mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \cap \{z_i\}^\perp$ and $i \leftarrow i + 1$. Go to Step 2.



How to find $\mathcal{F}$? - Facial Reduction

Theorem (The facial reduction theorem)

Suppose (CFP) *is feasible. There is a chain of faces*

$$\mathcal{F}_\ell \subsetneq \cdots \subsetneq \mathcal{F}_1 = \mathcal{K}$$

and vectors $(z_1, \ldots, z_{\ell-1})$ *such that:*

1. *For all* $i \in \{1, \ldots, \ell - 1\}$, *we have*

$$z_i \in \mathcal{F}_i^* \cap \mathcal{L}^\perp \cap \{a\}^\perp,$$
$$\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{z_i\}^\perp.$$

2. $\mathcal{F}_\ell \cap (\mathcal{L} + a) = \mathcal{K} \cap (\mathcal{L} + a)$ *and* $(\mathrm{ri}\, \mathcal{F}_\ell) \cap (\mathcal{L} + a) \neq \emptyset$.

L, M. Muramatsu and T. Tsuchiya.
Facial reduction and partial polyhedrality.
SIAM Journal on Optimization, 28(3), 2018 (http://arxiv.org/abs/1512.02549).

J. M. Borwein and H. Wolkowicz.
Regularizing the abstract convex program.
Journal of Mathematical Analysis and Applications, 83(2):495 – 530, 1981.

General strategy

Goal: We want to bound $\operatorname{dist}(x, (\mathcal{L}+a) \cap \mathcal{K})$ using $\operatorname{dist}(x, \mathcal{L}+a)$ and $\operatorname{dist}(x, \mathcal{K})$.

1. Find $\mathcal{F}$ such that
   a. $\mathcal{F} \cap (\mathcal{L}+a) = \mathcal{K} \cap (\mathcal{L}+a)$
   b. $(\operatorname{ri} \mathcal{F}) \cap (\mathcal{L}+a) \neq \emptyset$

   Therefore,

$$\operatorname{dist}(x, \mathcal{K} \cap (\mathcal{L}+a)) \leq \kappa_B(\operatorname{dist}(x, \mathcal{F}) + \operatorname{dist}(x, \mathcal{L}+a)), \quad \forall x \in B. \tag{1}$$

2. Upper bound $\operatorname{dist}(x, \mathcal{F})$ using $\operatorname{dist}(x, \mathcal{K})$ and $\operatorname{dist}(x, \mathcal{L}+a)$.
3. Plug the upper bound in (1).

Step 1 done!

Ingredient 2 - One-step Facial Residual Functions

Let

- $\mathcal{K}$: closed convex cone.
- $z \in \mathcal{K}^*$

Definition ($\mathbb{1}$-FRF for $\mathcal{K}$ and z)

If $\psi_{\mathcal{K},z} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

1. $\psi_{\mathcal{K},z}$ is nonnegative, monotone nondecreasing in each argument and $\psi(0,\alpha) = 0$ for every $\alpha \in \mathbb{R}_+$.
2. whenever $x \in \operatorname{span} \mathcal{K}$ satisfies the inequalities

$$\operatorname{dist}(x, \mathcal{K}) \leq \epsilon, \quad \langle x, z \rangle \leq \epsilon,$$

we have:

$$\operatorname{dist}(x, \mathcal{K} \cap \{z\}^{\perp}) \leq \psi_{\mathcal{K},z}(\epsilon, \|x\|).$$

Main result

Theorem (Error bound based on $\mathbb{1}$-FRF, Lindstrom, L., Pong)

Let $\mathcal{K}$ be a closed convex cone such that $\mathcal{K} \cap (\mathcal{L} + a) \neq \emptyset$. Let

$$\mathcal{F}_\ell \subsetneq \cdots \subsetneq \mathcal{F}_1 = \mathcal{K}$$

be a chain of faces of $\mathcal{K}$ together with $z_i \in \mathcal{F}_i^* \cap \mathcal{L}^\perp \cap \{a\}^\perp$ such that

$$(\mathcal{L} + a) \cap \operatorname{ri} \mathcal{F}_\ell \neq \emptyset.$$

and $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{z_i\}^\perp$ for every i. Let ψ_i be a $\mathbb{1}$-FRF for $\mathcal{F}_i$, z_i. Then, there is a suitable positively rescaled shift of the ψ_i, such that for every bounded B there are $\kappa > 0$, $M > 0$ such that

$$x \in B, \quad \operatorname{dist}(x, \mathcal{K}) \leq \epsilon, \quad \operatorname{dist}(x, \mathcal{L} + a) \leq \epsilon,$$

implies

$$\operatorname{dist}\,(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq \kappa(\epsilon + \varphi(\epsilon, M)),$$

where $\varphi = \psi_{\ell-1} \diamondsuit \cdots \diamondsuit \psi_1$, if $\ell \geq 2$. If $\ell = 1$, we let φ be the function satisfying $\varphi(\epsilon, \|x\|) = \epsilon$.

$$(f \diamondsuit g)(a, b) := f(a + g(a, b), b).$$

Main result

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$$x \in B, \quad \mathrm{dist}\,(x, \mathcal{K}) \leq \epsilon, \quad \mathrm{dist}\,(x, \mathcal{L} + a) \leq \epsilon,$$

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$$\mathrm{dist}\,(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq \kappa(\epsilon + \varphi(\epsilon, M)),$$

where $\varphi = \psi_{\ell-1} \diamondsuit \cdots \diamondsuit \psi_1$, if $\ell \geq 2$. If $\ell = 1$, we let φ be the function satisfying $\varphi(\epsilon, \|x\|) = \epsilon$.

$$(f \diamondsuit g)(a, b) := f(a + g(a, b), b).$$

The case of symmetric cones - L'21

- $\mathcal{K}$: symmetric cone (psd matrices, second order cone and etc)
- $\mathbb{1}$-FRF: $\psi_{\mathcal{F},z}(\epsilon, t) = \kappa\epsilon + \kappa\sqrt{\epsilon t}$

Suppose $(\mathcal{L} + a) \cap \mathcal{K} \neq \emptyset$.

There exists $\gamma \geq 0$ such that for every bounded B, there exists κ_B such that

$$\operatorname{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq \kappa_B (\operatorname{dist}(x, \mathcal{L} + a) + \operatorname{dist}(x, \mathcal{K}))^{(2^{-\gamma})}, \quad \forall\, x \in B$$

where γ is the number of facial reduction steps.

Reminder: $z \in \mathcal{F}^*$ and

$$x \in \operatorname{span} \mathcal{F}, \quad \operatorname{dist}(x, \mathcal{F}) \leq \epsilon, \quad \langle x, z \rangle \leq \epsilon,$$

implies

$$\operatorname{dist}(x, \mathcal{F} \cap \{z\}^{\perp}) \leq \psi_{\mathcal{F},z}(\epsilon, \|x\|).$$

p-cones

Let $\|\bar{x}\|_p := \sqrt[p]{|\bar{x}_1|^p + \cdots + |\bar{x}_n|^p}$.

- $\mathcal{K}_p^{n+1} := \{x = (x_0, \bar{x}) \in \mathbb{R}^{n+1} \mid x_0 \geq \|\bar{x}\|_p\}$
- Non-homogeneous, not self-dual even if the inner product is changed[1]
- $\mathbb{1}$-FRF: $\psi_{\mathcal{K}_p^{n+1},z}(\epsilon, t) = \kappa\epsilon + \kappa(\epsilon t)^{\alpha_z}$

$$\alpha_z := \begin{cases} \frac{1}{2} & \text{if } |\bar{z}|_0 = n, \\ \frac{1}{p} & \text{if } |\bar{z}|_0 = 1 \text{ and } p < 2, \\ \min\left\{\frac{1}{2}, \frac{1}{p}\right\} & \text{otherwise,} \end{cases}$$

For every bounded B, there exists κ_B such that

$$\mathrm{dist}\,(x, (\mathcal{L} + a) \cap \mathcal{K}_p^{n+1}) \leq \kappa_B (\mathrm{dist}\,(x, \mathcal{L} + a) + \mathrm{dist}\,(x, \mathcal{K}_p^{n+1}))^{\alpha_z}, \quad \forall\, x \in B$$

Tight result: the exponents cannot be improved.

[1] *The automorphism group and the non-self-duality of p-cones*, by Ito and L., JMAA'19

Least squares with p-norm regularization

$$\theta = \min_{x \in \mathbb{R}^n} \quad g(x) := \frac{1}{2}\|Ax - b\|^2 + \sum_{i=1}^{s} \lambda_i \|x_i\|_p, \qquad \text{(LS)}$$

Conic reformulation:

$$\begin{aligned} \min_{t,u,w,y,x} \quad & 0.5t + \sum_{i=1}^{s} \lambda_i y_i \\ \text{s.t.} \quad & Ax - w = b, \quad u = 1 \\ & (t, u, w) \in \mathcal{Q}_r^{m+2}, \quad (y_i, x_i) \in \mathcal{K}_p^{n_i+1}, \quad i = 1, \ldots, s. \end{aligned}$$

The **optimal set** is the intersection of

$$\mathcal{L} + a = \left\{ \mathrm{v} \;\middle|\; 0.5t + \sum_{i=1}^{s} \lambda_i y_i = \theta, u = 1, Ax - w = b \right\}$$

with the cone

$$\mathcal{K} = \mathcal{Q}_r^{m+2} \times \mathcal{K}_p^{n_1+1} \times \cdots \times \mathcal{K}_p^{n_s+1}.$$

$\Rightarrow$ g satisfies a Hölderian error bound condition with an **explicit** exponent[2]

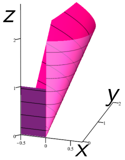
Theorem (LLP'21)

Let x^* be an optimal solution to (LS)*. Under a mild condition, g satisfies the KL property at x^* with exponent $1 - \alpha$, where $\alpha = \min\{0.5, 1/p\}$.*

[2]See *Kurdyka-Łojasiewicz Exponent via Inf-projection*, by Yu, Li, Pong, FoCM'21

The exponential cone

$$K_{\exp} := \left\{(x,y,z) \mid y > 0, z \geq y e^{x/y}\right\} \cup \{(x,y,z) \mid x \leq 0, z \geq 0, y = 0\}.$$



1. Applications to entropy optimization, logistic regression, geometric programming and etc.
2. Available in Alfonso, Hypatia, Mosek, SCS. https://docs.mosek.com/modeling-cookbook/expo.html.

V. Chandrasekaran, P. Shah
Relative entropy optimization and its applications.
Math. Program. 161, 1–32 (2017)

Error bounds for the exponential cone - LLP'20

$$\begin{aligned} \text{find} \quad & x \\ \text{subject to} \quad & x \in (\mathcal{L} + a) \cap K_{\exp} \end{aligned} \tag{CFP}$$

Four types of error bounds are possible:

- Lipschitzian error bound
- Hölderian error bound with exponent $1/2$
- **Entropic error bound**: for every bounded set B, there exists $\kappa_B > 0$

$$\operatorname{dist}(x, (\mathcal{L} + a) \cap K_{\exp}) \le \kappa_B \mathfrak{g}_{-\infty}(\max(\operatorname{dist}(x, \mathcal{L} + a), \operatorname{dist}(x, K_{\exp}))), \quad \forall x \in B.$$

- **Logarithmic error bound**: for every bounded set B, there exists $\kappa_B > 0$

$$\operatorname{dist}(x, (\mathcal{L} + a) \cap K_{\exp}) \le \kappa_B \mathfrak{g}_{\infty}(\max(\operatorname{dist}(x, \mathcal{L} + a), \operatorname{dist}(x, K_{\exp}))), \quad \forall x \in B,$$

where

$$\mathfrak{g}_{-\infty}(t) := -t \ln(t), \qquad \mathfrak{g}_{\infty}(t) := -\frac{1}{\ln(t)}, \qquad (\text{for } t \text{ small}).$$

The results above are **tight**.

Some remarks

- More stuff in the papers! Ex: direct products, techniques for obtaining FRFs, for proving tightness and so on.

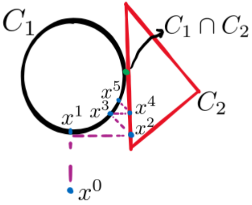
Scott B. Lindstrom; L and Ting Kei Pong
Error bounds, facial residual functions and applications to the exponential cone
arXiv:2010.16391

Scott B. Lindstrom; L and Ting Kei Pong
Tight error bounds and facial residual functions for the p-cones and beyond
arXiv:2109.11729

Other advertisement:

T. Liu and L.
Convergence analysis under consistent error bounds
arXiv:2008.12968

Error bounds provide information on the speed of algorithms



Lipschtizian error bound $\Longrightarrow$ **Linear** convergence :

$$\operatorname{dist}(x^k, C_1 \cap C_2) \leq M\theta^k$$

Hölderian error bound $\Longrightarrow$ **Sublinear** convergence:

$$\operatorname{dist}(x^k, C_1 \cap C_2) \leq Mk^{-\alpha}$$

J. M. Borwein, G. Li, and M. K. Tam.
Convergence rate analysis for averaged fixed point iterations in common fixed point problems.
SIAM Journal on Optimization, 27(1):1–33, 2017.

Consistent error bounds

$C_1, \ldots, C_m \subseteq \mathbb{R}^n$: closed convex sets

Definition (Consistent error bound functions)

$\Phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a (strict) **consistent error bound function** for $C_1, \ldots, C_m$ if:

(i)

$$\operatorname{dist}(x, \cap_{i=1}^m C_i) \le \Phi\left(\max_{1\le i\le m} \operatorname{dist}(x, C_i),\ \|x\|\right) \quad \forall\ x \in \mathbb{R}^n;$$

(ii) $\forall b \ge 0$, $\Phi(\cdot, b)$ is monotone (increasing) nondecreasing, right-continuous at 0 and $\Phi(0, b) = 0$;

(iii) $\forall a \ge 0$, $\Phi(a, \cdot)$ is monotone nondecreasing.

Fact: $\cap_{i=1}^m C_i \neq \emptyset \Rightarrow \exists \Phi$ a **strict consistent error bound function** for $C_1, \ldots, C_m$

T. Liu and L.
Convergence analysis under consistent error bounds
arXiv:2008.12968 **(Revised in 03/2022)**

Main convergence result

$C_1, \ldots, C_m \subseteq \mathbb{R}^n$: closed convex sets. $C = \cap_{i=1}^m C_i$.
Φ: *strict* consistent error bound function
$\{x^k\}$: sequence by some reasonable algorithm

For $\kappa > 0$ and $\delta > 0$ define $\phi_{\kappa,\Phi}(t) := \left(\Phi(\sqrt{t},\, \kappa)\right)^2$

$$\Phi_\kappa^\spadesuit(t) := \int_\delta^t \frac{1}{\phi_{\kappa,\Phi}^-(s)} ds$$

Then, the convergence of $\{x^k\}$ is either finite or $\exists \tau > 0$

$$\operatorname{dist}(x^k,\, C) \le \sqrt{(\Phi_{\widehat{\kappa}}^\spadesuit)^{-1}\Big(L - \tau k\Big)} \quad \forall\ k \ge 2\ell,$$

where $\widehat{\kappa} = \|x^0\| + 2\operatorname{dist}(0,\, C)$ and $L = \Phi_{\widehat{\kappa}}^\spadesuit(\operatorname{dist}^2(x^0,\, C))$.

Gee... that looks like hard.  How practical is that?

- For consistent error bound functions associated to Hölderian error bounds $(\Phi^{\spadesuit}_{\widehat{\kappa}})^{-1}$ has **closed form**.
- For other types of error bounds, there are upper bounds based on **Karamata theory**.

$f : (0, a] \to (0, \infty)$ is **regularly varying at** 0 with index ρ if

$$\lim_{x \to 0_+} \frac{f(\lambda x)}{f(x)} = \lambda^{\rho}, \quad \lambda > 0,$$

- Asymptotic behavior of regularly varying functions under taking integrals, inverses, powers is **very** well-understood.

N. H. Bingham, C. M. Goldie, and J. L. Teugels.
Regular Variation.
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Convergence rates under exotic error bounds - LL'20

Recall that

$$\operatorname{dist}(x^k, C_1 \cap C_2) \leq \sqrt{(\Phi_\kappa^\spadesuit)^{-1}\Big(L - \tau k\Big)} \quad \forall\ k \geq M,$$

Then,

- (i) **Entropic error bound**: The convergence rate is **almost linear**: for any $r > 0$, the following relations hold as $s \to +\infty$

$$\sqrt{((\Phi)_\kappa^\spadesuit)^{-1}(-s)} = o(s^{-r}), \qquad e^{-rs} = o\left(\sqrt{((\Phi)_\kappa^\spadesuit)^{-1}(-s)}\right).$$

- (ii) **Logarithmic** error bound: The convergence rate is logarithmic

$$\eta_1\left(\frac{1}{\ln(s)}\right) \leq \sqrt{(\Phi_\kappa^\spadesuit)^{-1}(-s)} \leq \eta_2\left(\frac{1}{\ln(s)}\right), \quad \forall\ s \geq N.$$

Blue and Red are upper and lower bounds, respectively

Amenable cones

Definition (Amenable cones)

$\mathcal{K}$ is **amenable** if for every face $\mathcal{F}$ of $\mathcal{K}$ there is $\kappa > 0$ such that

$$\mathrm{dist}\,(x, \mathcal{F}) \leq \kappa \mathrm{dist}\,(x, \mathcal{K}), \qquad \forall x \in \mathrm{span}\, \mathcal{F}.$$

- Symmetric cones (e.g., PSD cone) are amenable ($\kappa = 1$)
- Polyhedral cones are amenable
- Strictly convex cones are amenable. (p-cones, second order cones and so on)
- Amenability is preserved under linear isomorphisms

L, V. Roshchina and J. Saunderson
Amenable cones are particularly nice.
arxiv:2011.07745

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Hyperbolicity cones are amenable.
arxiv:2102.06359

L.
Amenable cones: error bounds without constraint qualifications.

Facial exposedness

$$\mathcal{F} \text{ is a face of } \mathcal{K} \stackrel{\text{def}}{\iff} \mathcal{F} \trianglelefteq \mathcal{K}$$

1. Projectionally exposed cone (BW'81) $\stackrel{\text{def}}{\iff}$ $\forall \mathcal{F} \trianglelefteq \mathcal{K}$ there exists a projection such that $P\mathcal{K} = \mathcal{F}$.
2. Amenable cones (L'21) $\stackrel{\text{def}}{\iff}$ $\forall \mathcal{F} \trianglelefteq \mathcal{K}$ there is $\kappa > 0$ such that
$$\operatorname{dist}(x, \mathcal{F}) \leq \kappa \operatorname{dist}(x, \mathcal{K}), \qquad \forall x \in \operatorname{span} \mathcal{F}.$$
3. Nice cone (P'07) $\stackrel{\text{def}}{\iff}$ $\forall \mathcal{F} \trianglelefteq \mathcal{K}, \quad \mathcal{F}^* = \mathcal{K}^* + \mathcal{F}^\perp$.
4. Facially exposed cone $\stackrel{\text{def}}{\iff}$
$\forall \mathcal{F} \trianglelefteq \mathcal{K}, \quad \exists z \in \mathcal{K}$, s.t. $\mathcal{F} = \mathcal{K} \cap \{z\}^\perp$.

Other curious types of cones:

1. Perfect cones (B'78) $\stackrel{\text{def}}{\iff}$ $\mathcal{K}$ is self-dual and every face $\mathcal{F} \trianglelefteq \mathcal{K}$ is self-dual over $\operatorname{span} \mathcal{F}$.
2. Devious cones  (TW'12) $\stackrel{\text{def}}{\iff}$ $\mathcal{K} + \operatorname{span} \mathcal{F}$ is not closed for $\{0\} \neq \mathcal{F} \trianglelefteq \mathcal{K}$.

A comparison table

		Exposed	Nice	Amenable	Projectionally
Preserved under	finite intersections	✓	✓	✓	?
	direct product	✓	✓	✓	✓
	injective linear image	✓	✓	✓	✓
Symmetric cones		✓	✓(CT'08)	✓	✓L'21
Homogeneous cones		✓	✓(CT'08)	✓LRS'20	?
Hyperbolicity cones		✓(R'05)	✓	✓LRS'21	?

- Facially exposed $\overset{\text{P'13}}{\Leftarrow}$ Nice $\overset{\text{L'21}}{\Leftarrow}$ **Amenable** $\overset{\text{EPBR}}{\Leftarrow}$ Projectionally exposed.
- There exists a 4D cone that is facially exposed but not nice (Vera, SIOPT'14).
- There exists a 4D cone that is nice but not amenable LRS'20
- In dimension 4 or less: Amenable $\Leftrightarrow$ Projectionally exposed. LRS'20

Open questions

- Is there an amenable cone that is not projectionally exposed? (dim $\mathcal{K} \geq 5$ must hold!)
- Which cones are projectionally exposed?

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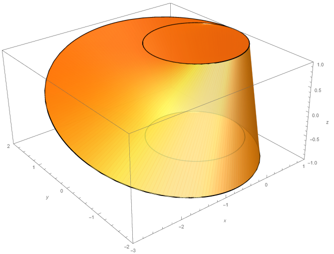

Figure: A 3D slice of a 4D convex cone that is nice but not amenable

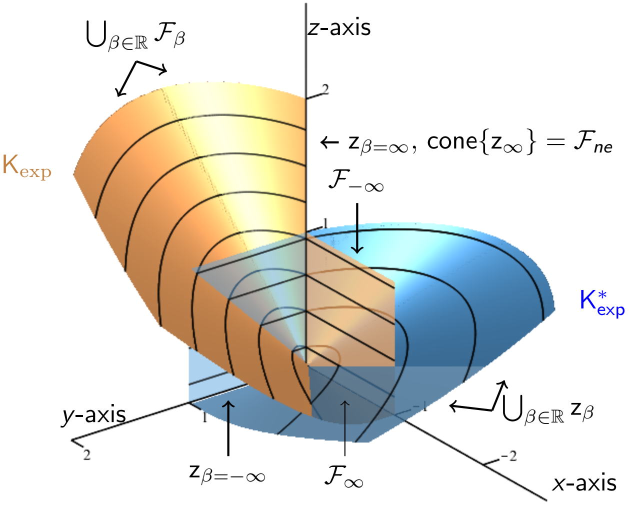

Figure: The exponential cone and its dual, with faces and exposing vectors labeled according to our index β.

Consequences for symmetric cone programming

$$\begin{aligned} \min_x \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \in \mathcal{K} \end{aligned}$$

Over a bounded set B:

For the feasible set:

- Under Slater: **Forward error** $= O($**Backward Error**$)$.
- Without Slater: **Forward error** $= O(($**Backward Error**$)^{2^{-\gamma}})$

For the optimal set:

- Strict complementarity holds: $x^* + s^* \in \operatorname{ri} \mathcal{K} \Leftrightarrow x^* \in \operatorname{ri}(\mathcal{K} \cap \{s^*\}^\perp)$
  - $\mathrm{Opt} = \{x \mid c^T x = \theta, Ax = b, x \in \mathcal{K}\}$ intersects $\operatorname{ri}(\mathcal{K} \cap \{s^*\}^\perp)$
  - Facial reduction finishes in 1 step.
- Under Strict complementarity:
  Forward error $= O(\sqrt{\textbf{Backward Error}})$