Completely solving general SDPs

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SDPs

$$\inf_{x} \quad \langle c, x \rangle \qquad \text{(P)}$$
$$\text{subject to} \quad \mathcal{A}x = b$$
$$x \in \mathcal{S}^n_+$$

$$\sup_{y} \quad \langle b, y \rangle \qquad \text{(D)}$$
$$\text{subject to} \quad c - \mathcal{A}^* y \in \mathcal{S}^n_+,$$

How to solve SDPs in *general*?

B. F. Lourenço, M. Muramatsu, and T. Tsuchiya,
Solving SDP completely with an interior point oracle
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Strange behaviour 1 - Duality gaps

$$\inf_x \quad -2x_{12} - 2x_{13} \ (\to 0,\ \to 0) \qquad \text{(P)}$$
$$\text{s.t.} \quad x_{11} = 0$$
$$-x_{22} - 2x_{13} = -1$$
$$x \in \mathcal{S}^3_+.$$

$$\sup_{t,s} \quad -s \ (\to -1) \qquad \text{(D)}$$
$$\text{s.t.} \quad \begin{pmatrix} t & 1 & s-1 \\ 1 & s & 0 \\ s-1 & 0 & 0 \end{pmatrix} \in \mathcal{S}^3_+$$

$\theta_D = -1$ and $\theta_P = 0$.

Strange behaviour 2 - Non-attainment

$$\begin{aligned} \sup_{t,s} \quad & -s \\ \text{s.t.} \quad & \begin{pmatrix} t & 1 \\ 1 & s \end{pmatrix} \in \mathcal{S}^2_+ \end{aligned} \tag{D}$$

Strange behaviour 3 - Weak infeasibility

$$\sup_{t,s} \quad t \qquad \text{(D)}$$

$$\text{s.t.} \quad \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{S}_+^2$$

- Let $V = \{c - \mathcal{A}^* y \mid y \in \mathbb{R}^n\}$
- In general, (D) feasible $\Rightarrow \operatorname{dist}(V, \mathcal{S}_+^n) = 0$
- Here, we have $\operatorname{dist}(V, \mathcal{S}_+^n) = 0$, but (D) is **infeasible**.

Multiple things at the same time

$$\sup_{y \in \mathbb{R}^8} \quad -y_4 - 2y_6 - 2y_7 \tag{D}$$

s.t.

$$\begin{pmatrix} y_1 & & & & & & & y_3 - 1 \\ & y_1 & & & & & & y_5 - 1 \\ & & y_2 & y_3 & & & & \\ & & y_3 & y_4 - y_5 & & & & \\ & & & & y_4 & -0.5y_8 + 0.5 & y_6 & \\ & & & & -0.5y_8 + 0.5 & y_8 & y_7 & \\ & & & & y_6 & y_7 & 0 & \\ y_3 - 1 & y_5 - 1 & & & & & & 0 \end{pmatrix} \in \mathcal{S}^8_+.$$

- $\theta_D = -1$, $\theta_P = 0$ and **neither are attained.**

How to solve SDPs in *general*?

1. IPMs? Some first order method? Probably won't work if there is positive duality gap or non-attainment
2. What if we try to regularize the SDP via *facial reduction* or something?
    1. It only fixes **one side of the problem**.

It is very hard to solve general SDPs! Even in low-dimensions and with apparently harmless data...

Ok, so which SDPs can we actually solve?

- If (P) and (D) **both** have interior points, then $\theta_P = \theta_D$ and are attained.
  - We have a decent chance of actually solving (P) and (D) with IPMs, augmented Lagrangian and etc.

The interior point oracle $\mathcal{O}_{\text{int}}$

Input:The problem data: $\mathcal{A}, b, c$. Both (P) and (D) must have interior points.
Output:A primal-dual optimal solution pair x^*, y^*.

The main result

Completely solving SDPs

Any SDP can be **completely solved** via polynomially (in *n*) many calls to $\mathcal{O}_{\text{int}}$

Completely solving (D) entails the following.

- Deciding feasibility and infeasibility.
  - In case of infeasibility, distinguishing between weak and strong infeasibility.
- Computing the optimal value
  - If attained, we also want an optimal solution.
  - If not, we compute an ϵ-optimal solution for any $\epsilon > 0$.
  - We also want to detect unboundedness.

Next we describe our tools: facial reduction and double facial reduction.

Facial Reduction Basics

$$\begin{array}{rl} \sup\limits_{y} & \langle b, y \rangle \\ \text{subject to} & c - \mathcal{A}^* y \in \mathcal{S}^n_+, \end{array} \tag{D}$$

Let $\mathcal{F}_{\mathrm{D}}$ denote the feasible slacks of (D), $\mathcal{F}_{\mathrm{D}} = \{S \in \mathcal{S}^n_+ \mid \exists y, c - \mathcal{A}^* y\}$

- If $\mathcal{F}_{\mathrm{D}}$ has no interior point of $\mathcal{S}^n_+$ then $\mathcal{F}_{\mathrm{D}}$ lies on a proper face of $\mathcal{S}^n_+$
- The smallest such face $\mathcal{F} \trianglelefteq \mathcal{S}^n_+$ contains $\mathcal{F}_{\mathrm{D}}$ and

$$\mathcal{F}_{\mathrm{D}} \cap \operatorname{ri} \mathcal{F} \neq \emptyset.$$

- Replacing $\mathcal{S}^n_+$ by $\mathcal{F}$ leads to a **smaller equivalent** problem that has an interior point!

J. M. Borwein and H. Wolkowicz.
Regularizing the abstract convex program.
Journal of Mathematical Analysis and Applications, 83(2):495 – 530, 1981.

More about facial reduction

- If $(c + \operatorname{range} \mathcal{A}^*) \cap \operatorname{ri} \mathcal{S}^n_+ = \emptyset$, we find a hyperplane $\{d\}^\perp$ that properly separates both, with $d \in \mathcal{S}^n_+$.
- Then, we replace $\mathcal{S}^n_+$ by $\mathcal{S}^n_+ \cap \{d\}^\perp$ and repeat.

Example:

$$\begin{pmatrix} t & 1 & s-1 \\ 1 & s & 0 \\ s-1 & 0 & 0 \end{pmatrix} \in \mathcal{S}^3_+$$

We can let $d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Facial reduction and $\mathcal{O}_{\text{int}}$

Theorem

Through $O(n)$ calls to $\mathcal{O}_{int}$ we can either detect that (D) is infeasible or find an equivalent SDP that has an interior point at the dual side.

Key idea: d can be found by solving by successively using $\mathcal{O}_{\text{int}}$ to solve

$$
\begin{aligned}
\inf_{x,t,w} \quad & t && (P_{\mathcal{K}}) \\
\text{subject to} \quad & -\langle c, x - te^* \rangle + t - w = 0 && (1) \\
& \langle e, x \rangle + w = 1 && (2) \\
& \mathcal{A}x - t\mathcal{A}e^* = 0 && (3) \\
& (x, t, w) \in \mathcal{K}^* \times \mathbb{R}_+ \times \mathbb{R}_+
\end{aligned}
$$

$$
\begin{aligned}
\sup_{y_1,y_2,y_3} \quad & y_2 && (D_{\mathcal{K}}) \\
\text{subject to} \quad & cy_1 - ey_2 - \mathcal{A}^* y_3 \in \mathcal{K} && (4) \\
& 1 - y_1(1 + \langle c, e^* \rangle) + \langle e^*, \mathcal{A}^* y_3 \rangle \geq 0 && (5) \\
& y_1 - y_2 \geq 0 && (6)
\end{aligned}
$$

with $\mathcal{K} = \mathcal{S}^n_+$, $\mathcal{K} = \mathcal{S}^n_+ \cap \{d_1\}^\perp$, $\mathcal{K} = \mathcal{S}^n_+ \cap \{d_2\}^\perp$ and so on.

The story so far

Suppose we wish to solve (D)

$$\sup_{y} \quad \langle b, y\rangle \qquad \text{subject to} \quad c - \mathcal{A}^* y \in \mathcal{S}_+^n, \tag{D}$$

From facial reduction we either detect infeasibility or obtain

$$\sup_{y} \quad \langle b, y\rangle \qquad \text{subject to} \quad c - \mathcal{A}^* y \in \mathcal{F}_{\min}^D. \tag{$\hat{\text{D}}$}$$

$$\inf_{x} \quad \langle c, x\rangle \qquad \text{subject to} \quad \mathcal{A}x = b, \quad x \in (\mathcal{F}_{\min}^D)^* \tag{$\hat{\text{P}}$}$$

where $\mathcal{F}_{\min}^D \subseteq \mathcal{S}_+^n \subseteq (\mathcal{F}_{\min}^D)^*$

($\hat{\text{D}}$) is equivalent to (D) and has (relative) interior points.

However we can not use $\mathcal{O}_{\text{int}}$ to solve ($\hat{\text{P}}$) and ($\hat{\text{D}}$) yet.

Double facial reduction

Idea: apply facial reduction to ($\hat{\text{P}}$).

$$\begin{aligned} \sup_y \quad & \langle b, y\rangle \\ \text{subject to} \quad & c - \mathcal{A}^* y \in \mathcal{S}^n_+, \end{aligned} \tag{D}$$

First FR

$$\begin{aligned} \sup_y \quad & \langle b, y\rangle \\ \text{subject to} \quad & c - \mathcal{A}^* y \in \mathcal{F}^D_{\min}. \end{aligned} \tag{$\hat{\text{D}}$}$$

$$\begin{aligned} \inf_x \quad & \langle c, x\rangle \\ \text{subject to} \quad & \mathcal{A}x = b \\ & x \in (\mathcal{F}^D_{\min})^* \end{aligned} \tag{$\hat{\text{P}}$}$$

Second FR

$$\begin{aligned} \sup_y \quad & \langle b, y\rangle \\ \text{subject to} \quad & c - \mathcal{A}^* y \in (\mathcal{F}^{\hat{\text{P}}}_{\min})^*. \end{aligned} \tag{D*}$$

$$\begin{aligned} \inf_x \quad & \langle c, x\rangle \\ \text{subject to} \quad & \mathcal{A}x = b \\ & x \in \mathcal{F}^{\hat{\text{P}}}_{\min} \end{aligned} \tag{P*}$$

The double FR theorem

$$\sup_y \quad \langle b, y\rangle \qquad (\hat{\mathrm{D}})$$
$$\text{subject to} \quad c - \mathcal{A}^* y \in \mathcal{F}^D_{\min}.$$

$$\inf_x \quad \langle c, x\rangle \qquad (\hat{\mathrm{P}})$$
$$\text{subject to} \quad \mathcal{A}x = b$$
$$x \in (\mathcal{F}^D_{\min})^*$$

$$\sup_y \quad \langle b, y\rangle \qquad (\mathrm{D}^*)$$
$$\text{subject to} \quad c - \mathcal{A}^* y \in (\mathcal{F}^{\hat{\mathrm{P}}}_{\min})^*.$$

$$\inf_x \quad \langle c, x\rangle \qquad (\mathrm{P}^*)$$
$$\text{subject to} \quad \mathcal{A}x = b$$
$$x \in \mathcal{F}^{\hat{\mathrm{P}}}_{\min}$$

We have $\mathcal{F}^D_{\min} \subseteq (\mathcal{F}^{\hat{\mathrm{P}}}_{\min})^*$, $\quad \mathcal{F}^{\hat{\mathrm{P}}}_{\min} \subseteq (\mathcal{F}^D_{\min})^*$.

Theorem

1. *(θ_D) is finite $\iff \mathcal{F}^{\hat{P}}_{\min} \neq \emptyset$. In this case,* (P*) *and* (D*) *both have relative interior points and*
$$\theta_D = \theta_{P*} = \theta_{D*}.$$
2. *$\theta_D = +\infty$ if and only if $\mathcal{F}^{\hat{P}}_{\min} = \emptyset$.*

The story so far - Part 2

$$\sup_y \quad \langle b, y \rangle \qquad \text{(D)}$$
$$\text{subject to} \quad c - \mathcal{A}^* y \in \mathcal{S}^n_+,$$

$$\sup_y \quad \langle b, y \rangle \qquad \text{(D*)}$$
$$\text{subject to} \quad c - \mathcal{A}^* y \in (\mathcal{F}^{\hat{\mathrm{P}}}_{\min})^*.$$

$$\inf_x \quad \langle c, x \rangle \qquad \text{(P*)}$$
$$\text{subject to} \quad \mathcal{A}x = b$$
$$x \in \mathcal{F}^{\hat{\mathrm{P}}}_{\min}$$

So far, we are able to

- Determine whether (D) is feasible or not.
- Compute θ_D and determine whether it is $+\infty$ or not.
- **Next steps**: obtaining optimal solutions.

Optimal solutions

If we know θ_D, we can solve the feasibility problem

$$\begin{aligned} \text{find} \quad & y \\ \text{subject to} \quad & c - \mathcal{A}^* y \in \mathcal{S}_+^n, \\ & \langle b, y \rangle = \theta_D \end{aligned} \tag{Feas}$$

in $O(n)$ calls to $\mathcal{O}_{\text{int}}$.

Unattained optimal solutions

$$\sup_y \quad \langle b, y\rangle \qquad (\hat{\mathrm{D}})$$
$$\text{subject to} \quad c - \mathcal{A}^* y \in \mathcal{F}^D_{\min}.$$

$$\inf_x \quad \langle c, x\rangle \qquad (\hat{\mathrm{P}})$$
$$\text{subject to} \quad \mathcal{A}x = b$$
$$x \in (\mathcal{F}^D_{\min})^*$$

$$\sup_y \quad \langle b, y\rangle \qquad (\mathrm{D}^*)$$
$$\text{subject to} \quad c - \mathcal{A}^* y \in (\mathcal{F}^{\hat{\mathrm{P}}}_{\min})^*.$$

$$\inf_x \quad \langle c, x\rangle \qquad (\mathrm{P}^*)$$
$$\text{subject to} \quad \mathcal{A}x = b$$
$$x \in \mathcal{F}^{\hat{\mathrm{P}}}_{\min}$$

Let y^* be an optimal solution do (D*)

The directions $\{d_1, \ldots, d_\ell\}$ obtained in the final FR and y^* can be used to construct y_ϵ:

$$c - \mathcal{A}^* y_\epsilon \in \mathcal{S}^n_+, \qquad \langle b, y_\epsilon\rangle \geq \theta_D - \epsilon.$$

Completely solving general SDPs

Using FR, double FR and $\mathcal{O}_{\text{int}}$ we can

- Detect feasibility and infeasibility.
    - in case of infeasibility: detect the type of feasbility.
- Compute the optimal value and an optimal solution if it exists.
- Compute ϵ-optimal solutions.

An example

$$
\begin{aligned}
\sup_{y \in \mathbb{R}^8} \quad & -y_4 - 2y_6 - 2y_7 \qquad\qquad \text{(D)} \\
\text{s.t.} \quad & \\
& \begin{pmatrix}
y_1 & & & & & & & y_3 - 1 \\
 & y_1 & & & & & & y_5 - 1 \\
 & & y_2 & y_3 & & & & \\
 & & y_3 & y_4 - y_5 & & & & \\
 & & & & y_4 & -0.5y_8 + 0.5 & y_6 & \\
 & & & & -0.5y_8 + 0.5 & y_8 & y_7 & \\
 & & & & y_6 & y_7 & 0 & \\
y_3 - 1 & y_5 - 1 & & & & & & 0
\end{pmatrix} \in \mathcal{S}^8_+ .
\end{aligned}
$$

- $\theta_D = -1$, $\theta_P = 0$ and **neither are attained.**

First FR

Let $\mathcal{S}_+^{r,n} := \left\{ \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{S}^n \mid U \in \mathcal{S}_+^r \right\}$.

- $\mathcal{F}_{\min}^D = \mathcal{S}_+^{6,8}$
- $(\mathcal{F}_{\min}^D)^* = \left\{ \begin{pmatrix} U & V \\ V & W \end{pmatrix} \in \mathcal{S}^8 \mid U \in \mathcal{S}_+^6 \right\}$

Second FR

Linear constraints of (P):

$$-x_{11} - x_{22} = 0 \quad - x_{33} = 0 \qquad -2x_{18} - 2x_{34} = 0 \quad - x_{44} - x_{55} = -1$$
$$-2x_{28} + x_{44} = 0 \quad - 2x_{57} = -2 \qquad -2x_{67} = -2 \quad x_{56} - x_{66} = 0$$

x is feasible for (P) if and only if $x \in \mathcal{S}^8_+$ and can be written as

$$\begin{pmatrix} 0_4 & & & & \\ & 1 & x_{56} & 1 & x_{58} \\ & x_{56} & x_{56} & 1 & x_{68} \\ & 1 & 1 & x_{77} & x_{78} \\ & x_{58} & x_{68} & x_{78} & x_{88} \end{pmatrix}$$

x is feasible for $(\hat{\mathrm{P}})$ if and only if $x \in (\mathcal{S}^{6,8})^*$, satisfies the linear equations above and can be written as

$$\begin{pmatrix} 0_3 & 0 & V_1 \\ 0 & U & V_2 \\ V_1^\top & V_2^\top & W \end{pmatrix},$$

where $U \in \mathcal{S}^n_+$.

Second FR - (cont.)

x is feasible for ($\hat{\mathrm{P}}$)$\Rightarrow$ x can be written as

$$\begin{pmatrix} 0_3 & 0 & V_1 \\ 0 & U & V_2 \\ V_1^\top & V_2^\top & W \end{pmatrix},$$

where $U \in \mathcal{S}^n_+$.

$$\mathcal{F}^{\hat{\mathrm{P}}}_{\min} = \left\{ \begin{pmatrix} 0_3 & 0 & V_1 \\ 0 & U & V_2 \\ V_1^\top & V_2^\top & W \end{pmatrix} \mid U \in \mathcal{S}^3_+ \right\}$$

$$(\mathcal{F}^{\hat{\mathrm{P}}}_{\min})^* = \left\{ \begin{pmatrix} Z_1 & Z_{12} & 0 \\ Z_{12} & U & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid U \in \mathcal{S}^3_+ \right\}$$

Second FR - (cont.)

$$\sup_{y\in\mathbb{R}^8} \quad -y_4 - 2y_6 - 2y_7 \qquad \text{(D*)}$$

s.t.

$$\begin{pmatrix} y_1 & & & & & & & y_3-1 \\ & y_1 & & & & & & y_5-1 \\ & & y_2 & y_3 & & & & \\ & & y_3 & y_4-y_5 & & & & \\ & & & & y_4 & -0.5y_8+0.5 & y_6 & \\ & & & & -0.5y_8+0.5 & y_8 & y_7 & \\ & & & & y_6 & y_7 & 0 & \\ y_3-1 & y_5-1 & & & & & & 0 \end{pmatrix} \in (\mathcal{F}^{\hat{\mathrm{P}}}_{\min})^*.$$

We have $y_3 = y_5 = 1$, $y_7 = y_6 = 0$. We can take $y_4 = y_5 = 1$, $y_8 = 1$ and $y_1 = y_2 = 0$.
The optimal value is attained, but is not feasible for (D).

Almost optimal solution

Let $y_1^1 = y_2^1 = 1$ and $y_3^1 = \cdots = y_8^1 = 0$ and let f^1 the corresponding matrix in $\operatorname{range} \mathcal{A}$, so that $f_1 = \begin{pmatrix} I_3 & 0 \\ 0 & 0_5 \end{pmatrix}$

Let

$$\hat{y} = (\hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4, \hat{y}_5, \hat{y}_6, \hat{y}_7, \hat{y}_8) = (1, 2, 1, 2, 1, 0, 0, 1),$$

so that $c - \mathcal{A}^* \hat{y} \in \operatorname{ri} \mathcal{F}^D_{\min}$.

Suppose $\epsilon = 0.1$. Let

$$\beta = \frac{\theta_D - \langle b, \hat{y} \rangle - \epsilon}{\theta_D - \langle b, \hat{y} \rangle} = 0.9.$$

Then,

$$\tilde{y} = \beta y_i^* + (1 - \beta)\hat{y} + 10 y^1$$

is an ϵ-optimal solution to (D).

$$\hat{s} = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 2 & 1 & & & & \\ & & 1 & 1 & & & & \\ & & & & 2 & & & \\ & & & & & 1 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix}$$

$$s^* = \begin{pmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & 0 & \color{blue}{1} & & & & \\ & & \color{blue}{1} & \color{blue}{0} & 0 & 0 & & \\ & & & 0 & 1 & 0 & & \\ & & & 0 & 0 & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix}$$

$$f_1 = \begin{pmatrix} I_3 & 0 \\ 0 & 0_5 \end{pmatrix}$$

Conclusion

1. A general SDP **can** be completely solved if you are only allowed to solve SDPs having interior points at the primal and dual sides.
2. More stuff in the paper! Ex: discussion on different types of infeasibility, in-depth analysis of double facial reduction and more.
3. The results are valid for **any closed convex cone**.

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