Error bounds for conic problems and an application to the exponential cone

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Feasibility problems over convex cones

Consider the following *feasibility problem over a convex cone* $\mathcal{K}$.

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & x \in (\mathcal{L} + a) \cap \mathcal{K} \end{array}$$

- $\mathcal{K}$: closed convex cone contained in some space $\mathcal{E}$.
- $\mathcal{L}$: subspace contained in $\mathcal{E}$.
- $a \in \mathcal{E}$.

($\mathcal{L} + a$ is an affine space)

Scott B. Lindstrom; L and Ting Kei Pong
Error bounds, facial residual functions and applications to the exponential cone
arXiv:2010.16391

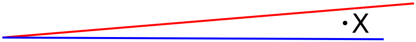
Motivation

Let $\|\cdot\|$ be the Euclidean norm and fix $x \in \mathcal{E}$.

$$\mathrm{dist}\,(x, \mathcal{L} + a) = \inf\{\|x - y\| \mid y \in \mathcal{L} + a\}$$
$$\mathrm{dist}\,(x, \mathcal{K}) = \inf\{\|x - y\| \mid y \in \mathcal{K}\}$$
$$\mathrm{dist}\,(x, (\mathcal{L} + a) \cap \mathcal{K}) = \inf\{\|x - y\| \mid y \in (\mathcal{L} + a) \cap \mathcal{K}\}$$

Fundamental question

Can we estimate $\mathrm{dist}\,(x, (\mathcal{L} + a) \cap \mathcal{K})$ using $\mathrm{dist}\,(x, \mathcal{L} + a)$ and $\mathrm{dist}\,(x, \mathcal{K})$?



- Convergence analysis often leads to this type of questions.

Hölderian error bounds

C_1, C_2: closed convex sets.
$C := C_1 \cap C_2$

Definition (Hölderian error bound)

C_1, C_2 satisfy a **Hölderian error bound** $\overset{\text{def}}{\iff}$ for every bounded set B there exist $\theta_B > 0$, $\gamma_B \in (0,1]$ such that

$$\mathrm{dist}\,(x,\ C) \le \theta_B(\mathrm{dist}\,(x, C_1) + \mathrm{dist}\,(x, C_2))^{\gamma_B} \quad \forall\ x \in B.$$

If $\gamma_B = \gamma \in (0,1]$ for all B, the bound is **uniform**. If the bound is uniform with $\gamma = 1$, we call it a **Lipschitzian** error bound.

Some known results:

- $\mathrm{ri}\, C_1 \cap \mathrm{ri}\, C_2 \neq \emptyset \Rightarrow$ Lipschitzian
- C_1, C_2 are polyhedral $\Rightarrow$ Lipschitzian (Hoffman's Lemma)

Conic results

- C_1: affine space, C_2: PSD cone $\Rightarrow$ Uniform Hölderian (Sturm's error bound)
- C_1: affine space, C_2: **amenable cone** $\Rightarrow$ generalized error bound holds (L'21)
- C_1: affine space, C_2: Symmetric cone $\Rightarrow$ Uniform Hölderian (L'21)

L.
Amenable cones: error bounds without constraint qualifications.
Mathematical Programming, 186:1–48, 2021.

J. F. Sturm.
Error bounds for linear matrix inequalities.
SIAM Journal on Optimization, 10(4):1228–1248, Jan. 2000.

Today's goals

- Prove error bounds for cones that might not be amenable
- Constraint qualifications are **forbidden**! 
- Apply our theory to the exponential cone.

Ingredient 1 - Error bounds under a constraint qualification

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & x \in (\mathcal{L} + a) \cap \mathcal{K} \end{array} \qquad \text{(CFP)}$$

Proposition (An error bound for when a face satisfying a CQ is known)

Let $\mathcal{F} \trianglelefteq \mathcal{K}$ be such that

- (a) $\mathcal{F}$ *contains* $\mathcal{K} \cap (\mathcal{L} + a)$
- (b) $(\operatorname{ri} \mathcal{F}) \cap (\mathcal{L} + a) \neq \emptyset$

Then, for every bounded set B, there exists $\kappa_B > 0$ such that

$$\operatorname{dist}(x, \mathcal{K} \cap (\mathcal{L} + a)) \leq \kappa_B (\operatorname{dist}(x, \mathcal{F}) + \operatorname{dist}(x, \mathcal{L} + a)), \qquad \forall x \in B.$$

It is not an error bound with respect to $\mathcal{L} + a$ and $\mathcal{K}$, but it is close.

Ingredient 2 - Facial Reduction

Theorem (The facial reduction theorem)

Suppose (CFP) *is feasible. There is a chain of faces*

$$\mathcal{F}_\ell \subsetneq \cdots \subsetneq \mathcal{F}_1 = \mathcal{K}$$

and vectors $(z_1, \ldots, z_{\ell-1})$ *such that:*

(i) *For all* $i \in \{1, \ldots, \ell - 1\}$, *we have*

$$z_i \in \mathcal{F}_i^* \cap \mathcal{L}^\perp \cap \{a\}^\perp,$$
$$\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{z_i\}^\perp.$$

(ii) $\mathcal{F}_\ell \cap (\mathcal{L} + a) = \mathcal{K} \cap (\mathcal{L} + a)$ *and* $(\mathrm{ri}\, \mathcal{F}_\ell) \cap (\mathcal{L} + a) \neq \emptyset$.

L, M. Muramatsu and T. Tsuchiya.
Facial reduction and partial polyhedrality.
SIAM Journal on Optimization, 28(3), 2018 (http://arxiv.org/abs/1512.02549).

J. M. Borwein and H. Wolkowicz.
Regularizing the abstract convex program.
Journal of Mathematical Analysis and Applications, 83(2):495 – 530, 1981.

Ingredient 3 - Facial Residual Functions

Let

- $\mathcal{K}$: closed convex pointed cone.
- $\mathcal{F}$: face of $\mathcal{K}$
- $z \in \mathcal{F}^*$

Fact:

$$\mathcal{F} \cap \{z\}^\perp = \mathcal{K} \cap \operatorname{span} \mathcal{F} \cap \{z\}^\perp.$$

Definition (Facial residual function for $\mathcal{F}$ and z with respect to $\mathcal{K}$)

If $\psi_{\mathcal{F},z} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

1. $\psi_{\mathcal{F},z}$ is nonnegative, monotone nondecreasing in each argument and $\psi(0,\alpha) = 0$ for every $\alpha \in \mathbb{R}_+$.
2. whenever $x \in \operatorname{span} \mathcal{K}$ satisfies the inequalities

$$\operatorname{dist}(x, \mathcal{K}) \leq \epsilon, \quad \langle x, z \rangle \leq \epsilon, \quad \operatorname{dist}(x, \operatorname{span} \mathcal{F}) \leq \epsilon$$

we have:

$$\operatorname{dist}(x, \mathcal{F} \cap \{z\}^\perp) \leq \psi_{\mathcal{F},z}(\epsilon, \|x\|).$$

Main result

Theorem (Error bound without amenable cones, Lindstrom, L., Pong)

Let $\mathcal{K}$ be a closed convex cone such that $\mathcal{K} \cap (\mathcal{L} + a) \neq \emptyset$. Let

$$\mathcal{F}_\ell \subsetneq \cdots \subsetneq \mathcal{F}_1 = \mathcal{K}$$

be a chain of faces of $\mathcal{K}$ together with $z_i \in \mathcal{F}_i^* \cap \mathcal{L}^\perp \cap \{a\}^\perp$ such that

$$(\mathcal{L} + a) \cap \operatorname{ri} \mathcal{F}_\ell \neq \emptyset.$$

and $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{z_i\}^\perp$ for every i. Let ψ_i be a facial residual function for $\mathcal{F}_i$, z_i. Then, after positive rescaling the ψ_i, for every bounded set B there are constants $\kappa > 0$, $M > 0$ such that if $x \in \operatorname{span} \mathcal{K} \cap B$ satisfies the inequalities

$$\operatorname{dist}(x, \mathcal{K}) \leq \epsilon, \quad \operatorname{dist}(x, \mathcal{L} + a) \leq \epsilon,$$

we have

$$\operatorname{dist}\, (x, (\mathcal{L} + a) \cap \mathcal{K}) \leq \kappa(\epsilon + \varphi(\epsilon, M)),$$

where $\varphi = \psi_{\ell-1} \diamondsuit \cdots \diamondsuit \psi_1$, if $\ell \geq 2$. If $\ell = 1$, we let φ be the function satisfying $\varphi(\epsilon, \|x\|) = \epsilon$.

$$(f \diamondsuit g)(a, b) := f(a + g(a, b), b).$$

Main result

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be a chain of faces of $\mathcal{K}$ together with $z_i \in \mathcal{F}_i^* \cap \mathcal{L}^\perp \cap \{a\}^\perp$ such that

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where $\varphi = \psi_{\ell-1} \diamondsuit \cdots \diamondsuit \psi_1$, if $\ell \geq 2$. If $\ell = 1$, we let φ be the function satisfying $\varphi(\epsilon, \|x\|) = \epsilon$.

$$(f \diamondsuit g)(a, b) := f(a + g(a, b), b).$$

Facial Residual Functions (FRFs) - Examples

- If $\mathcal{K}$ is a symmetric cone, then

$$\psi_{\mathcal{F},z}(\epsilon, t) = \kappa\epsilon + \kappa\sqrt{\epsilon t}$$

  is a FRF, for some $\kappa > 0$. (L'21)
  - Recovers Sturm's error bound when $\mathcal{K} = \mathcal{S}^n_+$.
- If $\mathcal{K}$ is polyhedral, then $\psi_{\mathcal{F},z}(\epsilon, \|x\|) = \kappa\epsilon$ is a FRF, for some $\kappa > 0$.

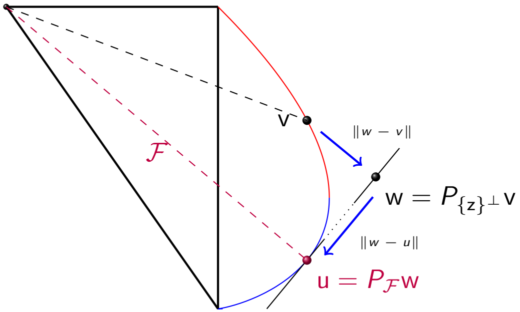
Reminder:

$$\mathrm{dist}\,(x, \mathcal{K}) \le \epsilon, \quad \langle x, z\rangle \le \epsilon, \quad \mathrm{dist}\,(x, \mathrm{span}\,\mathcal{F}) \le \epsilon$$

implies

$$\mathrm{dist}\,(x, \mathcal{F} \cap \{z\}^\perp) \le \psi_{\mathcal{F},z}(\epsilon, \|x\|).$$

FRFs without projection - LLP'21



$$\inf\left\{\frac{\|w-v\|^\alpha}{\|w-u\|}\right\} > 0 \quad \Rightarrow \quad \varphi(\epsilon,t) := \kappa_t\epsilon + \kappa_t\epsilon^\alpha \text{ is FRF}$$

$$\inf\left\{\frac{\mathfrak{g}(\|w-v\|)}{\|w-u\|}\right\} > 0 \quad \Rightarrow \quad \varphi(\epsilon,t) := \kappa_t\epsilon + \kappa_t\mathfrak{g}(2\epsilon) \text{ is FRF}$$

Facial residual functions and $\mathfrak{g}$-amenability

$\mathfrak{g} : \mathbb{R}_+ \to \mathbb{R}_+$: monotone nondecreasing function with $\mathfrak{g}(0) = 0$.

Definition ($\mathfrak{g}$-amenability)

$\mathcal{F} \trianglelefteq \mathcal{K}$ is $\mathfrak{g}$-*amenable* if for every bounded set B, there exists $\kappa > 0$ such that

$$\mathrm{dist}\,(x, \mathcal{F}) \leq \kappa \mathfrak{g}(\mathrm{dist}\,(x, \mathcal{K})), \quad \forall x \in (\mathrm{span}\, \mathcal{F}) \cap B.$$

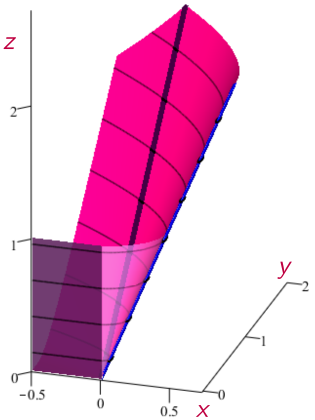
If all faces of $\mathcal{K}$ are $\mathfrak{g}$-amenable, then $\mathcal{K}$ is an $\mathfrak{g}$-*amenable cone*.

Suppose $\mathcal{K}^1$ and $\mathcal{K}^2$ are $\mathfrak{g}$-amenables

- There are calculus rules for the FRFs of $\mathcal{K}^1 \times \mathcal{K}^2$.
- A FRF of a **face** of $\mathcal{K}^1$ can be lifted to a FRF of the whole cone $\mathcal{K}^1$.
- Amenability is recovered when $\mathfrak{g} = |\cdot|$.
  - Amenable cones must be facially exposed.

The exponential cone

$$K_{\exp} := \left\{(x,y,z) \mid y > 0, z \geq ye^{x/y}\right\} \cup \{(x,y,z) \mid x \leq 0, z \geq 0, y = 0\}.$$



The exponential cone

$$K_{\exp} := \left\{(x,y,z) \mid y > 0, z \geq ye^{x/y}\right\} \cup \{(x,y,z) \mid x \leq 0, z \geq 0, y = 0\}\,.$$

$$K^*_{\exp} := \left\{(x,y,z) \mid x < 0, ez \geq -xe^{y/x}\right\} \cup \{(x,y,z) \mid x = 0, ez \geq 0, y \geq 0\}\,.$$

1. **Not exposed!** (So not amenable...)
2. Applications to entropy optimization, logistic regression, geometric programming and etc.
3. Available in Alfonso, Hypatia, Mosek. `https://docs.mosek.com/modeling-cookbook/expo.html`.

V. Chandrasekaran, P. Shah
Relative entropy optimization and its applications.
Math. Program. 161, 1–32 (2017)

The faces of the exponential cone

(a) exposed extreme rays (1D faces) parametrized by $\beta \in \mathbb{R}$:

$$\mathcal{F}_\beta := \left\{ \left(-\beta y + y, y, e^{1-\beta} y\right) \;\middle|\; y \in [0, \infty) \right\}. \qquad \text{(amenable)}$$

(b) an "exceptional" exposed extreme ray:

$$\mathcal{F}_\infty := \{(x, 0, 0) \mid x \leq 0\}. \qquad \text{(amenable)}$$

(c) a **non-exposed** extreme ray: $\mathcal{F}_{ne}$:

$$\mathcal{F}_{ne} := \{(0, 0, z) \mid z \geq 0\}. \qquad \text{($\mathfrak{g}$-amenable, not amenable)}$$

(d) a single 2D exposed face:

$$\mathcal{F}_{-\infty} := \{(x, y, z) \mid x \leq 0, z \geq 0, y = 0\}, \qquad \text{(amenable)}$$

where $\mathcal{F}_\infty$ and $\mathcal{F}_{ne}$ are the extreme rays of $\mathcal{F}_{-\infty}$.

The facial residual functions - 1D exposed faces

1. $\mathcal{F}_\beta := \left\{ \left(-\beta y + y, y, e^{1-\beta} y\right) \;\middle|\; y \in [0, \infty) \right\}$:

$$\psi_{\mathcal{K},z}(\epsilon, t) := \kappa\epsilon + \rho(t)\sqrt{\epsilon}$$

2. $\mathcal{F}_\infty := \{(x, 0, 0) \mid x \leq 0\}$:
   1. if $z_y > 0$, then:

$$\psi_{\mathcal{K},z}(\epsilon, t) := \rho(t)\epsilon$$

   2. if $z_y = 0$, then:

$$\psi_{\mathcal{K},z}(\epsilon, t) := \kappa\epsilon + \rho(t)\mathfrak{g}_\infty(\epsilon),$$

$$\mathfrak{g}_\infty(t) := \begin{cases} 0 & \text{if } t = 0, \\ -\frac{1}{\ln(t)} & \text{if } 0 < t \leq \frac{1}{e^2}, \\ \frac{1}{4} + \frac{1}{4}e^2 t & \text{if } t > \frac{1}{e^2}. \end{cases}$$

The facial residual functions - The 2D face and the non-exposed face

$$\mathfrak{g}_{-\infty}(t) := \begin{cases} 0 & \text{if } \; t = 0, \\ -t\ln(t) & \text{if } \; t \in \left(0, 1/e^2\right], \\ t + \frac{1}{e^2} & \text{if } \; t > 1/e^2. \end{cases}$$

1. $\mathcal{F}_{-\infty} := \{(x, y, z) \mid x \le 0, z \ge 0, y = 0\}$:

$$\psi_{\mathcal{K},z}(\epsilon, t) := \kappa\epsilon + \rho(t)\mathfrak{g}_{-\infty}(\epsilon)$$

2. $\mathcal{F}_{ne} := \{(0, 0, z) \mid z \ge 0\}$:

$$\psi_{\mathcal{F}_{-\infty},z}(\epsilon, t) := \sigma(t)\epsilon + \sigma(t)\mathfrak{g}_{-\infty}(\epsilon)$$

Error bound for problems over the exponential cone

$$\begin{aligned} &\text{find} \quad x \\ &\text{subject to} \quad x \in (\mathcal{L} + a) \cap K_{\exp} \end{aligned} \tag{CFP}$$

Let $z \in (K_{\exp})^* \cap \mathcal{L}^\perp \cap \{a\}^\perp$, $z \neq 0$. Let $\mathcal{F} = K_{\exp} \cap \{z\}^\perp$.

- $\mathcal{F} = \{0\}$: Lipschitzian error bound.
- $\mathcal{F} = \mathcal{F}_\beta$: a Hölderian error bound with exponent $1/2$.
- $\mathcal{F} = \mathcal{F}_\infty$, either a Lipschitzian or a log-type error bound holds depending on the exposing vector.
- $\mathcal{F} = \mathcal{F}_{-\infty}$, an **entropic error bound**: for every bounded set B, there exists $\kappa_B > 0$

$$\operatorname{dist}(x, (\mathcal{L} + a) \cap K_{\exp}) \leq \kappa_B \mathfrak{g}_{-\infty}(\max(\operatorname{dist}(x, \mathcal{L} + a), \operatorname{dist}(x, K_{\exp}))), \quad \forall x \in B.$$

The results above are **optimal**.

Strange error bounds

From the exponential cone we can:

- Obtain sets that **do not have** a Hölderian error bound, but have a logarithmic error bound:
  - Or, a function that does not have a KL exponent.

$$\mathcal{F}_{\infty} = K_{\exp} \cap \{z\}^{\perp},$$

  where $z = (0, 0, 1)$.
- Obtain sets that satisfy a Hölderian bound for all $\gamma \in (0, 1)$ but not $\gamma = 1$. Furthermore, the best error bound is an entropic one.
  - Or, a KL function whose exponent can be arbitrary close to $1/2$ but not $1/2$.

$$\mathcal{F}_{-\infty} = K_{\exp} \cap \{z\}^{\perp},$$

  where $z = (0, 1, 0)$.

Final remarks

- Much more stuff in the paper! Ex: direct products, techniques for obtaining FRFs and so on.

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arXiv:2010.16391

Other advertisement:

L; Vera Roshchina and James Saunderson
Amenable cones are particularly nice (**MS24**)
arXiv:2011.07745

T. Liu and L.
Convergence analysis under consistent error bounds (**MS69**)
arXiv:2008.12968

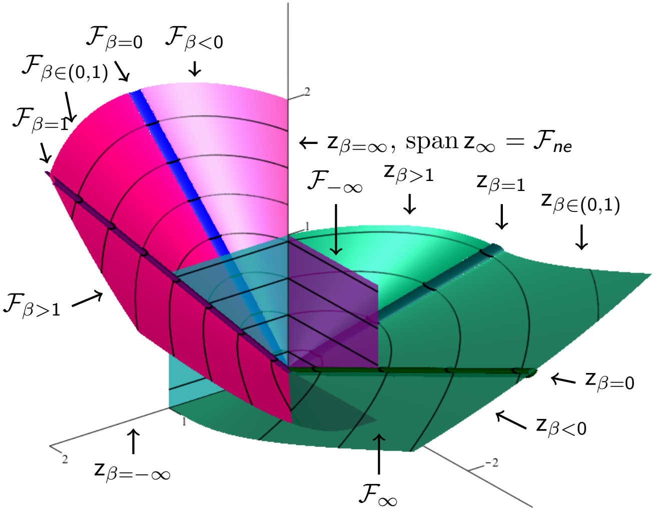

Figure: The exponential cone, its faces and exposing vectors